R-separation of variables for the conformally invariant Laplace equation

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Abstract

The conditions for R-separation of variables for the conformally invariant Laplace equation on an n-dimensional Riemannian manifold are determined and compared with the conditions for the additive separation of the null geodesic Hamilton-Jacobi equation. The case of 3-dimensions is examined in detail and it is proven that on any conformally flat manifold the two equations separate in the same coordinates.

Key words: R-separation, conformal separation, Laplace equation, conformal flatness, Stäckel separability
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1 Introduction

The subject of this paper is the study of R-separation of variables for the conformally invariant Laplace equation on an n-dimensional Riemannian manifold $(M, \mathbf{g})$. This equation may be written as

$$\mathbb{H}\psi := \Delta\psi + \frac{1}{4}\left(\frac{n-2}{n-1}\right) R_s \psi = 0, \tag{1}$$

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where

$$\Delta\psi := \sqrt{g}^{-1}\partial_i(\sqrt{g}g^{ij}\partial_j\psi) \tag{2}$$

is the Laplace-Beltrami operator and R_s is the Ricci scalar. A closely related problem is the study of additive separation of variables for the Hamilton-Jacobi equation for the null geodesics (in the pseudo-Riemannian case) namely

$$g^{ij}\partial_i W\partial_j W = 0. \tag{3}$$

An important property of both (1) and (3) is invariance under conformal transformation of the metric

$$\tilde{\mathbf{g}} = e^{2\sigma}\mathbf{g}. \tag{4}$$

It follows that if ψ is any solution of $\mathbb{H}\psi = 0$, then $\tilde{\psi} = e^{\frac{2-n}{2}}\psi$ is a solution of $\tilde{\mathbb{H}}\tilde{\psi} = 0$ on any conformally related manifold. Consequently, R-separability of the CI-Laplace equation is a conformally invariant property. This property is not shared by the Laplace equation

$$\Delta\psi = 0, \tag{5}$$

which is the equation most often studied in this regard [2,3,4]. Note that (5) is the equation usually considered as the extension of the ordinary Laplace equation,

$$\frac{\partial^2\psi}{\partial x_1^2} + \ldots + \frac{\partial^2\psi}{\partial x_n^2} = 0, \tag{6}$$

to a general non-flat Riemannian manifold $(M, \mathbf{g})$. However, we remark that both (1) and (5) reduce to the ordinary Laplace equation on any flat manifold.

In this paper we shall demonstrate the advantages of studying R-separability for the CI-Laplace equation continuing the work begun in [8]. In Section 2, based on the results of [4,1], we derive necessary and sufficient conditions for R-separation in the coordinates (q^i) in terms of the components of the contravariant metric tensor (g^{ij}). The precise result is given in Theorem 2.4. In Section 3 we consider the case $n = 3$. We first employ conformal invariance and Stäckel theory to derive the form of the contravariant metric in general conformally separable coordinates. We call a coordinate system *general* if it contains no conformally ignorable coordinate. We next obtain the contravariant metric in general R-separable coordinates in terms of an arbitrary fifth degree polynomial function of the coordinates q^i by imposing the compatibility condition of Theorem 2.4. It follows that R-separation of (1) occurs in general conformally separable coordinates if and only if the metric is conformally flat. We conclude this section by obtaining conditions for R-separation on the Laplace equation (5) on conformally flat spaces. In Section 4 we show that on the locally conformally flat 3-sphere $\mathbb{S}_3$ the Laplace equation (5) admits R-separation only if the coordinates are separable, while for the CI-Laplace equation there exist proper conformally separable coordinates allowing fixed

energy R-separation. This example provides a justification for studying R-separation for the conformally invariant equation.

For the flat case (in which the CI-Laplacian becomes the classical one) we recover the results given by Bôcher [2] and Boyer et al. [3] about the conformal factor such that the metric in general R-separable coordinates is flat and the transformation to Cartesian coordinates. Furthermore, we applied these results to provide CI-Laplace R-separable coordinates on other conformally flat manifolds. Section 5 contains the conclusion.

2 The CI-Laplace equation and R-separation

We define R-separation for the CI-Laplace equation, according to [4], as follows:

Definition 2.1 Multiplicative R-separation of a single second order partial differential equation (PDE) is the search for a solution ψ of the form

$$\psi = R(q^1, \ldots, q^n) \prod_i \phi_i(q^i, c_a) \qquad c_a \in \mathbb{R} \quad a = 1, \ldots, 2n-1; \tag{7}$$

satisfying the completeness condition

$$\operatorname{rank} \left[\frac{\partial}{\partial c_a} \left(\frac{\phi_i'}{\phi} \right) \ \frac{\partial}{\partial c_a} \left(\frac{\phi_i''}{\phi} \right) \right] = 2n-1, \qquad a = 1, \ldots, 2n-1, \quad i = 1, \ldots, n.$$

Remark 2.2 The completeness condition is equivalent to the fact that for any choice of the $2n-1$ values c_a there is a unique R-separated solution such that at a point $q_0 \in M$

$$\left(\frac{\phi_i'}{\phi}, \frac{\phi_j''}{\phi} \right)_{q_0} = (c_a) \qquad (i = 1, \ldots, n, \ j = 1, \ldots, n-1).$$

Moreover, the PDE splits into n separated ordinary differential equations (see [9,10]). The constants (c_a) are of two types: $n-1$ of them are separating constants, involved into the separated ordinary differential equations (ODE)s, the other n are integration constants, arising from the integration of the ODEs (see [4]).

The study of R-separation of the CI-Laplace equation, instead of the classical equation, seems more natural [8]. Indeed, the following property holds.

Proposition 2.3 *The existence of a complete R-separated solution of the CI-Laplace equation is a conformally invariant property that holds on the whole class of conformally related metrics.*

PROOF. Follows directly from (7) and the conformal invariance of (1). $\square$

In order to obtain necessary and sufficient conditions for the existence of a complete R-separated solution in a given coordinate system, first we transform R-separation into the multiplicative separation of a related PDE involving the function R: R-separated solutions of $\mathbb{H}\psi = 0$ correspond to multiplicatively separated solutions

$$\phi = \frac{\psi}{R} = \textstyle\prod_i \phi_i(q^i, c_a)$$

of

$$\Delta\phi + 2\nabla \ln R \cdot \nabla\phi + U\phi = 0,$$

where U is the modified potential

$$U := \frac{1}{4}\left(\frac{n-2}{n-1}\right) R_s + \frac{\Delta R}{R}. \tag{8}$$

By applying the techniques giving differential conditions for the R-separation of a single PDE [4] we arrive at:

Theorem 2.4 *Equation* (*1*) *admits R-separation in the coordinates* (q^i) *if and only if*

(1) the coordinates are orthogonal: $g_{ij} = 0, \quad i \neq j$;
(2) the contravariant components (g^{ii}) *satisfy the differential condition*

$$\frac{S_{ij}(g^{hh})}{g^{hh}} = \frac{S_{ij}(g^{kk})}{g^{kk}}, \qquad (\forall\ h,k,\ \forall\, i \neq j,\ i,j \text{ n.s.}) \tag{9}$$

where S_{ij} are second order operators, called Stäckel operators, *defined as*

$$S_{ij}(f) = \partial_i\partial_j f - \partial_i \ln|g^{jj}|\partial_j f - \partial_j \ln|g^{ii}|\partial_i f; \tag{10}$$

(3) the function R is (up to separated factors) a solution of

$$\partial_i \ln R = \tfrac{1}{2}\Gamma_i, \tag{11}$$

where $\Gamma_i = g^{hk}\Gamma_{hki}$;
(4) the modified potential U is a pseudo-Stäckel factor i.e., it is of the form $U = g^{hh} f_h(q^h)$ *for suitable functions* f_h.

PROOF. We apply the conditions for R-separation of the fixed energy R-separation of the Schrödinger equation

$$-\frac{\hbar^2}{2}\Delta\psi + (V-E)\psi = 0$$

given in [4] to equation (1), that is for $E = 0$ and $V = -\frac{\hbar^2}{8}\left(\frac{n-2}{n-1}\right) R_s$. $\square$

Remark 2.5 Orthogonal coordinates satisfying condition (9) are called *conformally separable* (see [1]), while orthogonal coordinates satisfying $S_{ij}(g^{hh}) = 0$ are said to be *simply separable*. The additive separation of variables for the null geodesic Hamilton-Jacobi equation in orthogonal coordinates,

$$g^{ii}(\partial_i W)^2 = 0, \tag{12}$$

and for the geodesic Hamilton-Jacobi equation

$$\tfrac{1}{2}g^{ii}(\partial_i W)^2 = E, \quad (E \in \mathbb{R}), \tag{13}$$

occurs if and only if the coordinates are conformally separable and simply separable, respectively. This fact shows an important link between Eq. (1) and Eq. (3).

Also in the Riemannian case, even if the null geodesics are trivial, the study of conformal separation can be applied effectively to the CI-Laplace equation. Indeed, as for the Laplace equation $\Delta\psi = 0$, we have that

Corollary 2.6 *A necessary condition for R-separation of the CI-Laplace equation (1) in a given coordinate system is that the null geodesic equation (3) is additively separable in the same coordinates.*

Remark 2.7 Two conditions equivalent to (9) are

- $\mathbf{g}$ is conformal to a metric which is separable for the geodesic Hamilton-Jacobi equation (13) in the same coordinates;
- there exists a Stäckel matrix S (that is a regular $n \times n$ matrix of which the elements of the ith-row depend only on q^i) such that [9]

  $$\frac{g^{ii}}{g^{jj}} = \frac{M_{in}}{M_{jn}}, \tag{14}$$

  where M_{in} is the minor of S obtained by eliminating the i-th row and the n-th column. We remark that the elements of the last column of the Stäckel matrix are not involved in (14).

Remark 2.8 The PDE system (11) determines the form of the factor R; the integrability conditions of the system are satisfied when the metric is conformally separable.

Remark 2.9 By inserting the form of R in the modified potential U we get

$$U = \frac{1}{4}g^{ii}(2\partial_i\Gamma_i - \Gamma_i^2) + \frac{1}{4}\left(\frac{n-2}{n-1}\right) R_s.$$

Two equivalent forms of the compatibility condition (4) are the following:

- the conformal metric $U^{-1}g^{hh}$ is a separable metric;
- condition $S_{ij}(g^{hh})U = S_{ij}(U)g^{hh}$ holds.

The conditions are formally the same as for the classical Laplace equation, except for the presence of the Ricci scalar in the modified potential. The condition that the coordinates are conformally separable holds for all conformally related metrics. Moreover, because of the term containing R_s in U, the compatibility condition is also satisfied (or not) on the whole class of conformally related metrics.

Proposition 2.10 *Let $\tilde{U}$ be the modified potential associated with $\tilde{g}^{hh} = e^{-2\sigma}g^{hh}$. Then $\tilde{U} = e^{-2\sigma}U$ and it is a pseudo-Stäckel factor if and only if U is.*

3 The three-dimensional case

The CI-Laplace equation on a three dimensional manifold is

$$\Delta\psi + \frac{1}{8}R_s\psi = 0. \tag{15}$$

Definition 3.1 A coordinate q^i is *conformally ignorable* if it appears in the conformal factor of the metric only, that is if $\partial_i(g^{hh}/g^{kk}) = 0$ for all h, k. We call a coordinate system *general* if it does not contain any conformally ignorable coordinate.

Up to a coordinate transformation of the form $\tilde{q}^i(q^i)$, a coordinate q^i is conformally ignorable if and only if ∂_i is a conformal Killing vector, that is an infinitesimal conformal symmetry. We restrict ourselves to general coordinate systems, leaving as a further development the analysis of the cases involving conformal symmetries.

The form of the general conformally separable coordinates in a three dimensional manifold is given in the following proposition (see [3])

Proposition 3.2 *In general conformally separable coordinates (q^i), the form of the (contravariant) metric on a 3-manifold is given by*

$$g^{ii} = Qh_i(q^i)(q^{i+2} - q^{i+1}) \qquad i = 1, \ldots, 3 \ (mod\ 3) \tag{16}$$

where Q is the conformal factor, and h_i three arbitrary functions of a single variable.

PROOF. Following [3], without loss of generality, we may choose S to be a 3×3 Stäckel matrix with third column set equal to unity:

$$S = \begin{bmatrix} \phi_1 & \psi_1 & 1 \\ \phi_2 & \psi_2 & 1 \\ \phi_3 & \psi_3 & 1 \end{bmatrix} \tag{17}$$

Then, we have

$$g^{11} = Q(\psi_3\phi_2 - \psi_2\phi_3), \quad g^{22} = Q(\psi_1\phi_3 - \psi_3\phi_1), \quad g^{33} = Q(\psi_2\phi_1 - \psi_1\phi_2). \tag{18}$$

In the general case, we can assume that none of the ψ_i and ϕ_i are identically null. Thus

$$g^{11} = Q\phi_2\phi_3\left(\frac{\psi_3}{\phi_3} - \frac{\psi_2}{\phi_2}\right), \; g^{22} = Q\phi_1\phi_3\left(\frac{\psi_1}{\phi_1} - \frac{\psi_3}{\phi_3}\right), \; g^{33} = Q\phi_1\phi_2\left(\frac{\psi_2}{\phi_2} - \frac{\psi_1}{\phi_1}\right). \tag{19}$$

By transforming each coordinate $\tilde{q}^i = \tilde{q}^i(q^i)$ and the conformal factor such that

$$\tilde{g}^{ii} \to \phi_i g^{ii} \qquad \tilde{Q} \to Q\phi_1\phi_2\phi_3,$$

we get $g^{ii} = Q(F_{i+2} - F_{i+1})$ with $F_i(q^i) = \dfrac{\psi_i}{\phi_i}$. If none of the F_i is a constant, then we can use them as coordinates; thus, we obtain

$$g^{11} = Qh_1(q^1)(q^3 - q^2), \quad g^{22} = Qh_2(q^2)(q^1 - q^3), \quad g^{33} = Qh_3(q^3)(q^2 - q^1). \qquad \square$$

Remark 3.3 If one of the elements of the Stäckel matrix is zero or one of the functions F_i is a constant, then, up to a coordinate transformation $\tilde{q}^i(q^i)$, one of the coordinates is conformally ignorable.

By Proposition 3.2 and Theorem 2.4 we arrive at

Theorem 3.4 *The form of the metric in general R-separable coordinates for the CI-Laplace equation is*

$$g^{ii} = QP(q^i) \cdot (q^{i+2} - q^{i+1}) \qquad i = 1, \ldots, 3 \; (mod\ 3) \tag{20}$$

where P is an arbitrary fifth-degree polynomial.

PROOF. Computing the modified potential U for the general conformal separable metric (16) and imposing the compatibility condition

$$S_{ij}(g^{hh})U = S_{ij}(U)g^{hh},$$

we get three additional differential conditions on the functions h_i that form a linear second order ODE system in three unknowns, whose solution is $h_i = P(q^i)$, where P is an arbitrary fifth-degree polynomial. $\square$

It is interesting to note that the compatibility condition has an intriguing geometrical interpretation.

Theorem 3.5 *On a 3-manifold, R-separation of the CI-Laplace equation occurs in general conformal separable coordinates if and only if the metric is conformally flat.*

PROOF. The conformal flatness conditions for a 3-dimensional Riemannian manifold are (see for instance [6])

$$R_{ijk} = \nabla_k R_{ij} - \nabla_j R_{ik} + \tfrac{1}{4}(g_{ik}\nabla_j R_s - g_{ij}\nabla_k R_s) = 0, \tag{21}$$

where ∇ is the covariant derivative and R_{ij} the covariant Ricci tensor. By imposing these conditions on the general conformal separable metric (16), we get three independent second order linear ODEs in the h_i which are equivalent to those allowing R-separation. Thus, the conformal flatness condition is equivalent to the compatibility condition for R-separation. $\square$

Remark 3.6 If one or more conformally ignorable coordinates appear, then being conformally flat is a sufficient but no longer a necessary condition for R-separation. Hence, in particular, equations (1) and (3) separate in the same orthogonal coordinates for all conformally flat 3-manifolds.

We can apply these results to the study of R-separation for the classical Laplace equation:

Theorem 3.7 *If a three dimensional manifold satisfies $R_s = 0$, then R-separation of the Laplace equation $\Delta\psi = 0$ occurs in general coordinates if and only if the manifold is conformally flat.*

PROOF. Since $R_s = 0$, the CI-Laplace equation and Laplace equation coincide. $\square$

Theorem 3.8 *On a conformally flat three dimensional manifold, R-separation of the Laplace equation $\Delta\psi = 0$ occurs in general conformally separable coordinates if and only if the Ricci scalar R_s satisfies the compatibility condition*

$$S_{ij}(g^{hh})R_s = S_{ij}(R_s)g^{hh}.$$

PROOF. Since the manifold is conformally flat, the CI-Laplace equation admits R-separation of variables in general conformally separable coordinates, that is the modified potential

$$U = \frac{\Delta R}{R} + \frac{R_s}{8}$$

satisfies $S_{ij}(g^{hh})U = S_{ij}(U)g^{hh}$. Then, the modified potential for the Laplace equation $U_L = \frac{\Delta R}{R}$ satisfies the same condition if and only if R_s does. $\square$

4 Applications and examples

In this section we apply the discussed techniques to examples on the sphere and other conformally flat three-dimensional manifolds.

Example 4.1 The 3-sphere $\mathbb{S}_3$ is a conformally flat 3-manifold with constant Ricci scalar $R_s > 0$. Hence, R-separation of the Laplace equation occurs only in those conformally separable coordinates such that

$$S_{ij}(g^{hh})R_s = S_{ij}(R_s)g^{hh} = 0,$$

since R_s is a constant. Thus, the metric components must satisfy $S_{ij}(g^{hh}) = 0$ and the coordinates are necessarily separable (see Remark 2.5). Due to this additional condition on the metric (20) the terms of degree five and four of the polynomial P must vanish and P reduces to a third degree polynomial. Moreover, by examining the conditions for the R-separation of the Helmholtz equation (see [7])

$$\Delta\psi = E\psi \qquad (E \in \mathbb{R}),$$

we have that, on $\mathbb{S}_3$, the Laplace equation,

$$\Delta\psi = 0,$$

admits R-separation only in the general separable coordinates such that R-separation of the Helmholtz equation occur.

Finally, we remark that on $\mathbb{S}_3$ the CI-Laplace equation (15) coincides with the Helmholtz equation for the fixed value of the energy

$$E = -\tfrac{1}{8}R_s.$$

Hence, locally there exist proper conformally separable coordinates allowing fixed energy R-separation of the Helmholtz equation for the value $E = -R_s/8$. For details see Example 4.4.

A fundamental example is the flat case, where the Laplace equation and the CI-Laplace equation are the same; it has been treated by several authors (see [2,11,9,3]).

Example 4.2 On the Euclidean three dimensional space $\mathbb{E}^3$ the CI-Laplace equation is $\Delta\psi = 0$. In order to determine the expression of general R-separable coordinates on $\mathbb{E}^3$ we need to compute the conformal factor Q such that the metric (20) is flat and the coordinate transformations from a Cartesian coordinate system. We restrict ourselves to the case where all roots of P are real and distinct (see [2] for the detailed analysis of all the possibilities). Let us denote by (q^1, q^2, q^3) the R-separable coordinates, (x^1, x^2, x^3) the Cartesian coordinates and by $e_1 < e_2 < e_3 < e_4 < e_5$ the five roots of the polynomial P. By using pentaspherical coordinates (see [2,11,3]) we can derive the following relations linking Cartesian coordinates to the R-separable ones (we adopt the same notation as in [3])

$$\begin{aligned}
\lambda \cdot x^1 &= \sqrt{\frac{(q^1-e_2)\cdot(q^2-e_2)\cdot(q^3-e_2)}{(e_2-e_1)\cdot(e_2-e_3)\cdot(e_2-e_4)\cdot(e_2-e_5)}} \\
\lambda \cdot x^2 &= \sqrt{\frac{(q^1-e_3)\cdot(q^2-e_3)\cdot(q^3-e_3)}{(e_3-e_1)\cdot(e_3-e_2)\cdot(e_3-e_4)\cdot(e_3-e_5)}} \\
\lambda \cdot x^3 &= \sqrt{\frac{(q^1-e_4)\cdot(q^2-e_4)\cdot(q^3-e_4)}{(e_4-e_1)\cdot(e_4-e_2)\cdot(e_4-e_3)\cdot(e_4-e_5)}}
\end{aligned} \tag{22}$$

where

$$\begin{aligned}
\lambda = &\sqrt{\frac{(q^1-e_1)\cdot(q^2-e_1)\cdot(q^3-e_1)}{(e_1-e_2)\cdot(e_1-e_3)\cdot(e_1-e_4)\cdot(e_1-e_5)}} \\
&+\sqrt{\frac{-(q^1-e_5)\cdot(q^2-e_5)\cdot(q^3-e_5)}{(e_5-e_1)\cdot(e_5-e_2)\cdot(e_5-e_3)\cdot(e_5-e_4)}}
\end{aligned} \tag{23}$$

and

$$e_1 < q^1 < e_2 < q^2 < e_3 < q^3 < e_4 < e_5.$$

It can be checked directly by expressing the Euclidean metric $\mathbf{g}_E$ with respect to the coordinates (q^i) that we obtain

$$\begin{aligned}
\mathbf{g}_E = \delta_{ij} dx^i \otimes dx^j &= \delta_{ij} \frac{-(q^i - q^{i+1})(q^i - q^{i+2})}{4P(q^i)\lambda^2} dq^i \otimes dq^j \\
&= \delta_{ij} \frac{\prod_h (q^h - q^{h+1})}{4P(q^i)\lambda^2 (q^{i+2} - q^{i+1})} dq^i \otimes dq^j,
\end{aligned}$$

which is of the form (20) with conformal factor Q_E given by

$$Q_E = \frac{4\lambda^2}{\prod_h (q^h - q^{h+1})}, \tag{24}$$

according to the formulas given in [2] and in [3] (where a factor of 4 seems to be missing).

Remark 4.3 The conformal metric

$$\tilde{g}_{ii} = \frac{(q^i - q^{i+1})(q^i - q^{i+2})}{P(q^i)}$$

is the general three-dimensional conformally flat metric allowing multiplicative separation of the Helmholtz equation computed by Eisenhart [5].

The formulas for the flat case can be adapted to a general conformally flat manifold $(M, \mathbf{g}_M)$. Since M is conformally flat, there exists a coordinate system (X^i) such that

$$\mathbf{g}_M = Q_{ME}^{-1} \sum_i dX^i \otimes dX^i,$$

where Q_{ME} is the conformal factor transforming $\mathbf{g}_M$ into the flat Euclidean metric $\mathbf{g}_E$. Then, if we formally replace (x^1, x^2, x^3) by (X^1, X^2, X^3) in the transformations (22) we obtain the coordinate transformations from (X^i) to the R-separable coordinates (q^i). Indeed, by inserting these relations in the metric $\mathbf{g}_M$, we have

$$\mathbf{g}_M = Q_{ME}^{-1} \sum_i dX^i \odot dX^i = Q_{ME}^{-1} Q_E^{-1} \sum_i [P(q^i) \cdot (q^{i+2} - q^{i+1}) dq^i \odot dq^i].$$

Hence, $Q_M = Q_{ME} Q_E$ is the conformal factor that transforms the general conformally flat metric (20) into a metric on the specific conformally flat manifold M. Then, in order to compute the conformal factor and the coordinate transformation, we only need to know the coordinates X^i on M corresponding to the Cartesian coordinates on $\mathbb{E}_3$.

In the following example we develop explicitly the case of $\mathbb{S}_3$.

Example 4.4 Let (X^1, X^2, X^3) be stereographic coordinates on $\mathbb{S}_3$, considered as a submanifold of $\mathbb{E}_4$. They are related to the Cartesian coordinates $(x^1, \ldots, x^4)$ of $\mathbb{E}_4$ by the following equations

$$\begin{aligned} x^a &= \frac{2r^2 X^a}{r^2 + \sum_{i=1}^3 (X^i)^2} \qquad a = 1, \ldots, 3 \\ x^4 &= r - \frac{2r^3}{r^2 + \sum_{i=1}^3 (X^i)^2} \end{aligned}$$

where r is the radius of the sphere. The components of the metric of $\mathbb{S}_3$ in the coordinates (X^i) are (see also [6])

$$g_{ii} = \frac{4r^4}{(r^2 + \sum_{i=1}^{3}(X^i)^2)^2}.$$

Hence, the function $Q_{SE} = (r^2 + \sum_{i=1}^{3}(X^i)^2)^2/4r^4$ is the conformal factor relating $\mathbb{S}_3$ to $\mathbb{E}_3$. Then,

$$Q_S = \frac{\lambda^2[(r^2 + \sum_{i=1}^{3}(X^i(q^1, q^2, q^3))^2)^2]}{r^4 \prod_h (q^h - q^{h+1})}$$

is the conformal factor which makes (20) the metric of $\mathbb{S}_3$. The coordinates (q^1, q^2, q^3), related to the stereographic coordinates (X^1, X^2, X^3) by

$$\begin{aligned}
\lambda \cdot X^1 &= \sqrt{\tfrac{(q^1-e_2)\cdot(q^2-e_2)\cdot(q^3-e_2)}{(e_2-e_1)\cdot(e_2-e_3)\cdot(e_2-e_4)\cdot(e_2-e_5)}},\\
\lambda \cdot X^2 &= \sqrt{\tfrac{(q^1-e_3)\cdot(q^2-e_3)\cdot(q^3-e_3)}{(e_3-e_1)\cdot(e_3-e_2)\cdot(e_3-e_4)\cdot(e_3-e_5)}},\\
\lambda \cdot X^3 &= \sqrt{\tfrac{(q^1-e_4)\cdot(q^2-e_4)\cdot(q^3-e_4)}{(e_4-e_1)\cdot(e_4-e_2)\cdot(e_4-e_3)\cdot(e_4-e_5)}},
\end{aligned}$$

with λ given by (23), are coordinates on $\mathbb{S}_3$ in which R-separation of the equation

$$\Delta\psi = \frac{3}{4r^2}\psi.$$

occurs for the fixed value of the energy determined by the radius of the sphere.

5 Conclusions

The examples show the effectiveness of the concept of fixed energy R-separation also to cases with non-zero energy and justifies the study of the CI-Laplacian. A direction for further research lies in the cases of metrics with one or more conformal symmetries, by using the techniques discussed. Another interesting open problem is the discussion for a general Riemannian manifold of the geometric interpretation of the compatibility condition 4 from Theorem 2.4.

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